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Invariance Principle for a Solid-on-Solid Interface Model

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A stochastic process describing the behavior of the solid-on-solid interface in a strip of width L is studied. The invariant and reversible measure for the process is the Gibbs state with Hamiltonian $H = \sum |\eta(x) - \eta(x+1)|$. Under "free" boundary conditions, we show that the height of the moving interface at any site converges, when suitable renormalized, to Brownian motion with a diffusion coefficient proportional to L^{-1}.

KEY WORDS: Solid-on-solid stochastic model.

1. INTRODUCTION

The statistical mechanics of the interface between two different fluids or materials has been studied in recent years.[11,14,15,8,5] The solid-on-solid model[13] has been reviewed by Fröhlich *et al.*[8] In this model the interface is a function η: $\mathbf{Z}^{d-1} \to \mathbf{Z}$. The Gibbs state corresponds to the Hamiltonian $\sum_{|x-y|=1} |\eta(x) - \eta(y)|$, with x, y in Λ_L, a square box of side $2L+1$ contained in $\mathbf{Z}^{d-1}$. A roughening transition occurs for two-dimensional interfaces: when $d-1=2$, the interface is rigid for sufficiently low temperatures and fluctuates logarithmically for high temperatures.[9,10] For more than three dimensions the interface remains rigid.[3] In two dimensions $|\eta(x) - \eta(y)|$ behaves like $|x-y|^{1/2}$ as a consequence of the fact that, under the Gibbs measure, $\{\eta(x) - \eta(x+1)\}_{x \in \Lambda_L}$ is a family of independent random variables. Gallavotti,[14] and Aizenman[1] show a similar behavior of the interface for the (more difficult) two-dimensional Ising model.

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In this paper I construct a stochastic process having as reversible measure the Gibbs state of the solid-on-solid model described above. For $d=2$ and *fixed* L, under free (or periodic) boundary conditions, I prove in Theorem 2.1 that, appropriately renormalized, the interface behaves rigidly as nondegenerate Brownian motion. More precisely, if the position of the interface at time t at site x is denoted $\eta_t(x)$, we prove that, as $\varepsilon \to 0$, the process $\varepsilon\eta_{\varepsilon^{-2}t}(0)$ converges to nondegenerate Brownian motion with diffusion coefficient (proportional to) $|\Lambda_L|^{-1}$ and that under this renormalization the difference between the interface at any two different sites converges in probability to zero. The result is less satisfactory in dimension $d \geqslant 3$: I am able to prove convergence to Brownian motion, but do not know how to prove that the diffusion coefficient is positive. Herbert Spohn showed to me that this result is immediate in the continuous case (see the discussion in Section 5).

In the infinite-volume case, in any dimension, under the hypothesis of existence of the infinite-volume Gibbs state, I show that the interface at a given site behaves as degenerate Brownian motion (Theorem 4.1). This result is only partially significant for $d=2$: for low temperatures in $d=3$ and for any temperature in $d>3$, one can expect the interface to be a stationary process in time, even without space renormalization.

Invariance principles for interfaces have been obtained for one-dimensional models. Galves and Presutti[12] showed that the interface of a one-dimensional supercritical contact process satisfies a central limit theorem. De Masi *et al.*[6] have proven an analogous result for the asymmetric simple exclusion process. Recently Schinazi[16] proved an invariance principle for the interface of a critical, reversible, semiinfinite, nearest particle system.

2. THE MODEL

The two-dimensional model is defined in an infinite strip C_L of base

$$\Lambda_L = \{-L, ..., L\} \subset \mathbf{Z}$$

given by

$$C_L = \Lambda_L \times \mathbf{Z} = \{(x, z): x \in \Lambda_L, z \in \mathbf{Z}\}$$

The strip C_L is occuppied by two phases separated by a line. The assumption of the solid-on-solid (SOS) model is that this line has no overhangs, and hence can be defined by a function $\eta: \Lambda_L \to \mathbf{Z}$. The interface state space is

$$\mathbf{X} = \{\eta: \Lambda_L \to \mathbf{Z}\}$$

and the Gibbs measure π_L with reciprocal temperature β is defined by

$$\mu_L(\eta) = e^{-\beta H(\eta)} \tag{2.1}$$

where the Hamiltonian H is given by

$$H(\eta) = \sum_{x \in \Lambda_L} |\eta(x) - \eta(x+1)|$$

Here and in the sequel we adopt the convention that if $y \notin \Lambda_L$, but $|x-y| = 1$ for some $x \in \Lambda_L$, then $\eta(y) \equiv \eta(x)$. This is what we have called "free boundary conditions." The results of this section hold also for periodic boundary conditions.

The total mass given by μ_L to $\mathbf{X}$ is infinity, but μ_L induces a probability measure in a subset of $\mathbf{X}$. Consider the following equivalence relation in $\mathbf{X}$: η is equivalent to ξ iff there exists a $k \in \mathbf{Z}$ such that $\eta(x) = \xi(x) + k$ for all $x \in \Lambda_L$. Identify each η such that $\eta(0) = 0$, with its equivalent class. In that way the space of the equivalence classes is isomorphic to

$$\mathbf{X}_0 = \mathbf{X} \cap \{\eta : \eta(0) = 0\}$$

We call $\tilde{\mu}_L$ the probability measure induced by μ_L in $\mathbf{X}_0$. That is,

$$\tilde{\mu}_L(\eta) := \frac{\mu_L(\eta)}{\mu_L(\mathbf{X}_0)} \tag{2.2}$$

This is well defined because $\mu_L(\mathbf{X}_0)$ is finite. When no confusion arises, we write $\tilde{\mu}$ instead of $\tilde{\mu}_L$.

Define the continuous-time Markov chain η_t on $\mathbf{X}$ with transition probability function given by

$$\begin{aligned} P(\eta_{t+\delta} = \eta^{x+} \mid \eta_t = \eta) &= \delta c(\eta, \eta^{x+}) + o(\delta) \\ P(\eta_{t+\delta} = \eta^{x-} \mid \eta_t = \eta) &= \delta c(\eta, \eta^{x-}) + o(\delta) \\ P(\eta_{t+\delta} = \zeta \mid \eta_t = \eta) &= o(\delta), \qquad \text{if} \quad \zeta \neq \eta^{x\pm} \end{aligned} \tag{2.3a}$$

where $c(\cdot, \cdot)$ are nonnegative functions and the configurations $\eta^{x+}, \eta^{x-} \in \mathbf{X}$ are defined by

$$\eta^{x\pm}(w) = \begin{cases} \eta(w) \pm 1 & \text{if } w = x \\ \eta(w) & \text{otherwise} \end{cases} \tag{2.3b}$$

In order to have μ as reversible measure, the transition rates $c(\cdot, \cdot)$ must satisfy the detailed balance condition

$$\mu(\eta)\, c(\eta, \eta^{x+}) = \mu(\eta^{x+})\, c(\eta^{x+}, \eta)$$

that is,

$$\frac{c(\eta, \eta^{x+})}{c(\eta^{x+}, \eta)} = \frac{\exp[-\beta \sum |\eta^{x+}(y) - \eta^{x+}(y+1)|]}{\exp[-\beta \sum |\eta(y) - \eta(y+1)|]} \tag{2.4a}$$

$$= \frac{\exp \beta[1\{\eta(x) < \eta(x+1)\} + 1\{\eta(x) < \eta(x-1)\}]}{\exp \beta[1\{\eta^{x+}(x) > \eta^{x+}(x+1)\} + 1\{\eta^{x+}(x) > \eta^{x+}(x-1)\}]} \tag{2.4b}$$

where $1\{\cdot\}$ is the indicator function of the set $\{\cdot\}$. An immediate choice for $c(\cdot, \cdot)$ is

$$c(\eta, \eta^{x+}) = \exp \beta[1\{\eta(x) < \eta(x+1)\} + 1\{\eta(x) < \eta(x-1)\}]$$

$$c(\eta, \eta^{x-}) = \exp \beta[1\{\eta(x) > \eta(x+1)\} + 1\{\eta(x) > \eta(x-1)\}]$$

but this is not the only one. We can impose another condition: we ask that the rate of increasing the interface at site x be a linear function of the number of neighbors y of x for which $\eta(y) > \eta(x)$. In other words, defining

$$h_x(\eta) = \sum_{y: |x-y|=1} 1\{\eta(y) > \eta(x)\} \tag{2.5a}$$

we ask that

$$c(\eta, \eta^{x+}) = bh_x(\eta) + c \tag{2.6a}$$

for some constants b and c. Analogously, if

$$k_x(\eta) = \sum_{y: |x-y|=1} 1\{\eta(y) < \eta(x)\} \tag{2.5b}$$

we ask that

$$c(\eta, \eta^{x-}) = bk_x(\eta) + c \tag{2.6b}$$

Now, substituting (2.5) in (2.4), we obtain that (2.6) is compatible with the detailed balance condition (2.4) if the constants b and c satisfy the following relation:

$$\frac{e^{\beta h}}{e^{\beta(2-h)}} = \frac{hb + c}{(2-h)b + c}$$

where we have used that $k_x(\eta^{x+}) = 2 - h_x(\eta)$. Since h can only assume values 0, 1, 2, we obtain that (2.4) and (2.6) are compatible for all $c > 0$ and b satisfying

$$b = c\left(\frac{e^{2\beta} - 1}{2}\right) \tag{2.7}$$

We have proven the following lemma:

Lemma 2.1. Let μ be the Gibbs state with inverse temperature $\beta > 0$ and Hamiltonian $H = \sum |\eta(x) - \eta(x+1)|$. Then there exists a process having μ as reversible measure with rates given by Eq. (2.6). The relationship between the inverse temperature and the coefficients b and c of Eq. (2.6) is given by Eq. (2.7).

Let η_t be the Markov chain in $\mathbf{X}$ defined in Eq. (2.3) with rates given by Eqs. (2.6)–(2.7). We prove the following:

Theorem 2.1. Fix $x \in \Lambda_L$. Let $\eta_t(x)$ be the height of the interface at site x at time t. Then:

(a) For any initial configuration $\eta \in \mathbf{X}$, the process $\varepsilon\eta_{\varepsilon^{-2}t}(x)$ converges weakly, as $\varepsilon \to 0$, to Brownian motion with diffusion coefficient D.

(b) The diffusion coefficient is proportional to $|\Lambda_L|^{-1}$. More precisely,

$$D = \frac{1}{|\Lambda_L|^2} \sum_{\eta \in \mathbf{X}_0} \tilde{\mu}(\eta) \sum_{y \in \Lambda_L} 2[c + b1\{\eta(y) \neq \eta(y+1)\}]$$

where $\tilde{\mu}$ is the probability Gibbs state obtained when the interface at the origin is fixed to be zero [see Eq. (2.2)].

(c) For each pair $x, y \in \Lambda_L$, $\varepsilon\eta_{\varepsilon^{-2}t}(x) - \varepsilon\eta_{\varepsilon^{-2}t}(y)$ converges in probability to zero.

We prove this theorem in the next section. Theorem 2.1 states that for fixed L, $0 < \beta < \infty$, the interface moves rigidly as a nondegenerate Brownian motion. For zero temperature ($\beta = \infty$) the rates are as in Eq. (2.6) with any $b \geqslant 0$ and $c = 0$, and the diffusion coefficient is zero. For infinite temperature the situation is different. In this case $b = 0$ and $c > 0$. The interface at different sites undergoes independent random walks at total rate $2c$. In the limit as $\varepsilon \to 0$ we obtain independent Brownian motions with diffusion coefficient $2c$.

3. PROOF OF THEOREM 2.1

A key role in the proof of Theorem 2.1 is played by the process ξ_t, "the interface as seen from an observer localized on the interface at the origin," defined by

$$\xi_t = \eta_t - \eta_t(0) \tag{3.1}$$

where the configuration $\eta - \eta(0) \in \mathbf{X}_0 = \mathbf{X} \cap \{\eta\colon\ \eta(0) = 0\}$ is defined by $(\eta - \eta(0))(x) = \eta(x) - \eta(0)$.

The Gibbs state $\tilde{\mu}$ on $\mathbf{X}_0$ defined in Eq. (2.2) can be written

$$\tilde{\mu}(\xi) = z^{-1} \exp\left\{-\beta \sum_{x \in \Lambda_L} |\xi(x) - \xi(x+1)|\right\} \tag{3.2}$$

where z is the normalization [so as to have $\sum \tilde{\mu}(\eta) = 1$]. One checks that the probability measure $\tilde{\mu}$ is reversible (therefore invariant) for the continuous-time Markov chain ξ_t on the denumerable space $\mathbf{X}_0$. Hence ξ_t is a positive recurrent Markov chain on $\mathbf{X}_0$. Since η_t is ergodic (the probability of going from one to another state in any fixed time is strictly positive for all states), $\tilde{\mu}$ is the unique invariant measure for this process.

Introduce the processes

$$M_t = \sum_{x \in \Lambda_L} \eta_t(x) \tag{3.3a}$$

and

$$\bar{M}_t = \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \eta_t(x) \tag{3.3b}$$

We interpret M_t as the total mass of one of the fluids (which can be negative) normalized to be zero when the configuration of the interface is identically zero. On the other hand, $\bar{M}_t$ is the mean value of the interface at time t.

Sketch of the proof of Theorem 2.1. We first show that both M_t and $\bar{M}_t$ are square-integrable ergodic martingales with respect to the filtration generated by ξ_t, $t \geqslant 0$. If the initial configuration of ξ_t is distributed accordingly to the reversible measure $\tilde{\mu}$, then M_t and $\bar{M}_t$ have stationary increments. We can then apply an invariance principle for martingales. The limit diffusion coefficients are easily bounded above and below. To conclude the proof, we observe that, when the initial measure of the process is $\tilde{\mu}$, $\bar{M}_t - \eta_t(0)$ is a random variable, whose distribution does not depend on t. Hence, as $\varepsilon \to 0$, $\varepsilon \bar{M}_{\varepsilon^{-2}t} - \varepsilon \eta_{\varepsilon^{-2}t}(0)$ goes to zero in probability.

Lemma 3.1. Both the total mass process M_t and the mean value process $\bar{M}_t$ are ergodic square-integrable martingales with respect to the filtration generated by $\{\eta_t \colon t \geqslant 0\}$. Furthermore, if the initial configuration η is distributed accordingly to $\tilde{\mu}$—the invariant measure for the process ξ_t—then both M_t and $\bar{M}_t$ have stationary increments.

Proof. Given a configuration $\eta \in \mathbf{X}$, the rate of increasing M_t by one is equal to

$$\sum_{x \in \Lambda_L} [b(1\{\eta(x) > \eta(x+1)\} + 1\{\eta(x) > \eta(x-1)\}) + c]$$
$$= \sum_{x \in \Lambda_L} [b(1\{\eta(x) < \eta(x+1)\} + 1\{\eta(x) < \eta(x-1)\}) + c]$$

which is the rate of decreasing M_t by one. In this way we have proved that M_t is a martingale and so is $\bar{M}_t$. This is the place where the choice of the rates (2.6) is crucial.

Since the distribution of the increments of M_t, for $t \geq s$, depends only on ξ_s and $\tilde{\mu}$ is stationary measure for ξ_s, then M_t has stationary increments. The ergodicity follows from the fact that ξ_t itself is an ergodic Markov chain.

To prove that it is square integrable, it suffices then to show that $\lim_{\delta \to 0}(EM_\delta^2/\delta)$ is finite, where E is the expectation of the process ξ_t obtained when the initial configuration is distributed according to $\tilde{\mu}$,

$$\begin{aligned}
\lim_{\delta \to 0} \frac{EM_\delta^2}{\delta} &= \sum_\eta \tilde{\mu}(\eta) \sum_{x \in \Lambda_L} [c + b(1\{\eta(x) > \eta(x+1)\} + 1\{\eta(x) > \eta(x-1)\}) \\
&\quad + c + b(1\{\eta(x) < \eta(x+1)\} + 1\{\eta(x) < \eta(x-1)\})] \\
&= \sum_\eta \tilde{\mu}(\eta) \sum_{x \in \Lambda_L} 2[c + b1\{\eta(x) \neq \eta(x+1)\}] \qquad (3.4)
\end{aligned}$$

which is clearly bounded above by $2(c+b)|\Lambda_L|$. This concludes the proof of Lemma 3.1.

Theorem 3.1. For any fixed initial configuration $\eta \in \mathbf{X}$, the processes $\varepsilon M_{\varepsilon^{-2}t}$ and $\varepsilon \bar{M}_{\varepsilon^{-2}t}$ converge in distribution to Brownian motion with positive diffusion coefficients D_M and $D_{\bar{M}}$, respectively. These are given by

$$D_M = \sum_\eta \tilde{\mu}(\eta) \sum_{x \in \Lambda_L} 2[c + b1\{\eta(x) \neq \eta(x+1)\}] \qquad (3.5)$$

and

$$D_{\bar{M}} = \frac{1}{|\Lambda_L|} D_M \qquad (3.6)$$

Proof. As a consequence of Lema 3.1, we can apply the invariance principle for martingales, which implies the theorem. We use the almost sure version as stated in ref. 4, Section 1. Since $\tilde{\mu}$ gives positive mass to each configuration of $\mathbf{X}_0$, we are able to substitute "almost all" for "all $\eta \in \mathbf{X}_0$." But any $\eta \in \mathbf{X}$ is a translation of some configuration in $\mathbf{X}_0$. The translation invariance of the dynamics and the renormalization allow us to substitute "all $\eta \in \mathbf{X}_0$" for "all $\eta \in \mathbf{X}$." The expressions for the diffusion coefficients [Eqs. (3.5) and (3.6)] are consequence of Eq. (3.4).

Proof of Theorem 2.1. Part (a). This follows from part (c) and Theorem 3.1: If for all $x, y \in \Lambda_L$, $\varepsilon\eta_{\varepsilon^{-2}t}(x) - \varepsilon\eta_{\varepsilon^{-2}t}(y)$ converges in probability to zero, so does $\varepsilon\eta_{\varepsilon^{-2}t}(x) - \varepsilon\bar{M}_{\varepsilon^{-2}t}$. Since, by Theorem 3.1, $\varepsilon\bar{M}_{\varepsilon^{-2}t}$ converges to Brownian motion with diffusion coefficient $D_{\bar{M}}$, so does $\varepsilon\eta_{\varepsilon^{-2}t}(y)$ for all $y \in \Lambda_L$. [We have also proven part (b).] Part (c). If the initial measure is $\tilde{\mu}$, for each pair of sites x, y of Λ_L, $\eta_t(x) - \eta_t(y) = \xi_t(x) - \xi_t(y)$ is a random variable whose distribution is independent of t. Then $\varepsilon\eta_{\varepsilon^{-2}t}(x) - \varepsilon\eta_{\varepsilon^{-2}t}(y)$ converges to zero in probability.

4. THE INFINITE-VOLUME CASE

The infinite-volume, d-dimensional model is defined in the state space

$$\mathbf{X} = \{\eta: \mathbf{Z}^{d-1} \to \mathbf{Z}\}$$

We consider the Gibbs state $\tilde{\mu}$ defined on $\mathbf{X}_0 = \mathbf{X} \cap \{\eta(0) = 0\}$ consistent with the conditional probabilities:

$$\tilde{\mu}(\xi(x) = j(x) \mid \xi(y) = j(y),\ y \neq x) = z^{-1} \exp\left\{-\beta \sum_{y: |x-y|=1} |j(x) - j(y)|\right\} \tag{4.1}$$

where $j(0) \equiv 0$ and $j(y)$ for $y \neq 0$ are integer numbers and z is a renormalizing constant. This is defined as the thermodynamic limit, as $L \to \infty$, of the $\tilde{\mu}_L$, defined as in Eqs. (2.2) and (3.2). The existence and unicity of the limit are immediate for $d = 2$, since by independence of $\{\eta(x) - \eta(x+1)\}_x$ under $\tilde{\mu}_L$, one constructs the measure explicitly. For $d \geqslant 3$ one must show some compactness argument.[7,8] In the sequel we assume the existence of the thermodynamic limit.

As in the finite case, we construct a stochastic process η_t with pregenerator Ω defined on bounded, cylindrical functions f by

$$\Omega f(\eta) = \sum_{x \in \mathbf{Z}} \{c(x, \eta, +)[f(\eta^{x+}) - f(n)] + c(x, \eta, -)[f(\eta^{x-}) - f(\eta)]\} \tag{4.2}$$

For the definitions of $\eta^{\pm}$ see Eq. (2.3). The rates $c(\cdot, \cdot, \pm)$ are imposed to be bounded, positive functions of x and to depend on η only through

$1\{\eta(x)>\eta(y)\}$, $1\{\eta(x)<\eta(y)\}$ for y such that $|y-x|=1$. The existence of a process η_t with semigroup $S(t)$ generated by Ω follows from the fact that the process $\eta(t)$ can be expressed as a function of a spin system in d dimensions with bounded rates depending only on the nearest neighbor spins.[8] We refer to Liggett[18] for the construction of the spin system.

In order to have $\tilde{\mu}$ as reversible measure, we ask the rates to satisfy

$$\frac{c(x,\eta,+)}{c(x,\eta^{x+},-)}=\frac{\exp\{-\beta\sum|\eta(x)+1-\eta(y)|\}}{\exp\{-\beta\sum|\eta(x)-\eta(y)|\}}$$

where the sums run over the set $\{y: |x-y|=1\}$. We can choose, for instance,

$$\begin{aligned} c(x,\eta,+)&=\exp\beta h_x(\eta)\\ c(x,\eta,-)&=\exp\beta k_x(\eta)\end{aligned} \tag{4.3}$$

where k and h are defined as in Eq. (2.5). As in the finite case [see Eq. (3.1)], we define

$$\xi_t=\eta_t-\eta_t(0)$$

Theorem 4.1. Let η_t be the process with generator given by (4.2) with rates gives by (4.3) and initial distribution $\tilde{\mu}$ given by (4.1). Then, as $\varepsilon\to 0$, the process

$$X_\varepsilon(t)=:\varepsilon\eta_{\varepsilon^{-2}t}(0) \tag{4.4}$$

converges to degenerate Brownian motion. That is, it converges in distribution to the process with trajectories concentrated in the constant 0. Furthermore,

$$D_\infty=:\lim_{t\to\infty}\frac{E(\eta_t(0)^2)}{t}=0 \tag{4.5}$$

where the expectation is taken with respect to the process ξ_t with initial measure $\tilde{\mu}$.

Remark. In the one-dimensional case $(d=2)$. Theorem 4.1 states that, as expected, $\lim D_L=D_\infty=0$. The result is not satisfactory. It just states that the normalization in Eq. (4.4) yields a trivial result. We conjecture that in order to get a nontrivial limit one must rescale space by $\varepsilon^{1/2}$ instead of ε: we expect that $\varepsilon^{1/2}\eta_{\varepsilon^{-2}t}$ converges, as $\varepsilon\to 0$, to a nondegenerate, normal random variable. This conjecture arises when one looks at the process of the differences $\eta_t(x)-\eta_t(x+1)$, $x\in\mathbf{Z}$, which is conservative in the sense that $\sum_x [\eta_t(x) - \eta_t(x+1)]$ is constant in time. This is analogous, then, to a lattice gas. In fact, the relationship between the simplest model of a stochastic lattice gas (nearest neighbor simple exclusion) and a special case of the SOS model has been studied in ref. 5 in a finite box. This relation can be also established in the infinite volume. In that case it is easy to see that the height of the interface at time t at the origin is equal to J_t, the total current of simple exclusion particles through the origin in the time interval $[0, t]$. It is known that the right renormalization for the current is $\varepsilon^{1/2}$: $\varepsilon^{1/2} J_{\varepsilon^{-2}t}$ converges to a nondegenerate normal random variable.[2] The same result can presumably be obtained by looking at the fluctuation fields[15] and applying an argument given by Spohn[17] (Remark in Section 2). In dimensions $d \geqslant 3$ the situation is different: the interface is rigid at low temperatures when $d = 3$ and for any temperature when $d > 3$. In this case the result of Theorem 4.1 is even less satisfactory, as one can expect the process $\eta_t(0)$ to be stationary without space rescaling.

Proof. The convergence to Brownian motion is a consequence of the results of ref. 4. The process $\eta_t(0)$ is antisymmetric and adapted to ξ_t, which has $\tilde{\mu}$ as reversible measure and is ergodic. Then, $X_\varepsilon(t)$ converges to Brownian motion with diffusion coefficient given by D_∞ defined in Eq. (4.5). In order to prove that this limit vanishes, take $\Lambda_L = \{-L,..., L\}^{d-1}$, and define

$$R_{L,t} = \sum_{x \in \Lambda_L} \eta_t(x) \tag{4.6}$$

and

$$\bar{R}_{L,t} = \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \eta_t(x) \tag{4.7}$$

These two processes are again antisymmetric and adapted to ξ_t, and hence with the same renormalization converge to Brownian motion with diffusion coefficients $D_{R,L}$ and $D_{\bar{R},L}$, respectively. Further, for all $L \geqslant 1$,

$$D_{\bar{R},L} = |\Lambda_L|^{-2} D_{R,L} \tag{4.8a}$$

and

$$D_{\bar{R},L} = D_\infty \tag{4.8b}$$

as follows from the fact that, under $\tilde{\mu}$, the differences $|\eta_t(x) - \eta_t(y)|$ are random variables whose distribution is independent of t and hence go to zero in probability when renormalized.

The next step is to prove that

$$D_{\bar{R},L} \leqslant \text{const}/|\Lambda_L| \tag{4.9}$$

In order to prove (4.9), we prove that $D_{R,L} \leqslant \text{const} \times |\Lambda_L|$ and then use (4.8). Let $\mathscr{F}_t$ be the σ-algebra generated by $\{\xi_s : s \leqslant t\}$, then,

$$R_{L,t} = N_{L,t} + \int_0^t \varphi_L(\xi_s)\, ds \tag{4.10}$$

where $N_{L,t}$ is a stationary, square-integrable martingale with respect to the filtration $\mathscr{F}_t$, $t \geqslant 0$, and φ_L is given by[4]

$$\begin{aligned} \varphi_L(\xi_t) &= \lim_{\delta \to 0} \frac{E[R_{L,t+\delta} - R_{L,t} \mid \mathscr{F}_t]}{\delta} \\ &= \sum_{x \in \Lambda_L} [c(x, \xi_t, +) - c(x, \xi_t, -)] \end{aligned} \tag{4.11}$$

where E is the expectation with respect to the process ξ_t with initial measure $\tilde{\mu}$. Furthermore,

$$EN_{L,t}^2 = tD_{N,L}$$

where

$$\begin{aligned} D_{N,L} &= \lim_{\delta \to 0} \frac{EN_{L,\delta}^2}{\delta} \\ &= \lim_{\delta \to 0} \frac{ER_{L,\delta}^2}{\delta} \\ &= \sum_{x \in \Lambda_L} \tilde{\mu}[c(x, \xi, +) + c(x, \xi, -)] \\ &\leqslant |\Lambda_L|\, 2B \end{aligned} \tag{4.12}$$

where B is the upper bound of the rates.

On the other hand, Eqs. (4.10) and (4.11) and Theorem 2.3 of ref. 4 imply that

$$D_{R,L} = D_{N,L} - 2 \int_0^\infty \tilde{\mu}[\varphi \hat{S}(t) \varphi]\, dt \tag{4.13}$$

where $\hat{S}(t)$ is the semigroup corresponding to the process ξ_t which is self-adjoint in $L^2(\tilde{\mu})$, by reversibility. Then the integral on the right-hand side of Eq. (4.13) is nonnegative and

$$D_{R,L} \leqslant D_{N,L} \leqslant 2B|\Lambda_L|$$

by Eq. (4.12), which concludes the proof of Eq. (4.9) and the theorem is shown.

5. DISCUSSION

Using the techniques developped in ref. 4 one can prove parts (a) and (c) of Theorem 2.1 in any dimensions and for any choice of rates $c(\cdot, \cdot)$ satisfying the detailed balance condition (2.4). The problem in proving (b) is that only in one dimension and for our choice of the rates is the process a martingale. In the general case, the diffusion coefficient satisfies an equation like (4.13).[4] When the process is a martingale, the second term vanishes and the first term gives the diffusion coefficient. The first term can be calculated at time $t=0$: it is the quadratic variation of the martingale. In general, when the process is not a martingale, it is not obvious how to prove that the two terms do not cancel. The second term is always negative because, by reversibility, $S(t)$ is self-adjoint in $L^2(\mu)$.

One could consider a process satisfying (a) and (b) of Lemma 2.8 in $d \geqslant 2$. In this case the total mass process is still a martingale. Unfortunately this process does not have a reversible measure, as was noticed by Paola Calderoni. The problem left in this case is to prove that the process ξ_t has an invariant measure. This would show that the martingale has stationary increments, and the invariant principle would follow.

For other Hamiltonians $H = \sum A|\eta(x) - \eta(y)|$, where A is a nonnegative, nondecreasing function, we can always construct a process η_t having as reversible measure μ constructed with this Hamiltonian, in a square box of radius L. Since $E\varphi^2 < \infty$, one can apply ref. 4 and show (a) and (c) of Theorem 2.1. In any case, it seems that (b) is a difficult problem. For the infinite-volume case even the existence of the process is not clear unless the rates are bounded.

Finally, let us discuss briefly the continuous model. In this case the state space is the set of functions η: $\Lambda_L \to \mathbb{R}$. The equation for $\eta_t(x)$ is assumed to be[15]

$$d\eta_t(x) = \frac{1}{2}\, dt \sum_{y:|y-x|=1} U'(|\eta_t(x) - \eta_t(y)|) + \sigma dW(x, t)$$

where $U(r)$, $r \in \mathbb{R}^+$, is a symmetric, smooth, convex $(U'' > 0)$ function.

$W(x, t)$ is a standard Brownian motion. The stationary measures are the Gibbs states given by (formally)

$$\mu(d\eta) = \prod_{x \in \Lambda_L} d\eta(x) \exp -\beta H(x, \eta)$$

where

$$H(x, \eta) = \sum_{y:\,|y-x|=1} U(|\eta(x) - \eta(y)|)$$

and $\beta = 2/\sigma^2$. Herbert Spohn showed me that if we define (as in the discrete case) $M_t = \sum_{x \in \Lambda_L} \eta_t(x)$, the drift terms cancel and we obtain that $M_t = \sum W(x, t)$, which is just the ordinary Brownian motion. From this, the analog of Theorem 2.1 follows for the continuous case.

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